Universal Statistics of Inviscid Burgers Turbulence in Arbitrary Dimensions

Ali Naji[1] and Shahin Rouhani[1,2]

[1]Department of Physics, Sharif University of Technology,
Tehran, P.O.Box: 11365-9161, Iran
[2]Institute for Studies in Theoretical Physics and Mathematics,
Tehran, P.O.Box: 19395-5531, Iran
e-mail: naaji@physics.sharif.ac.ir

February 6, 2008

Abstract

We investigate the non-perturbative results of multi-dimensional forced Burgers equation coupled to the continuity equation. In the inviscid limit, we derive the exact exponents of two-point density correlation functions in the universal region in arbitrary dimensions. We then find the universal generating function and the tails of the probability density function (PDF) for the longitudinal velocity difference. Our results exhibit that in the inviscid limit, density fluctuations affect the master equation of the generating function in such a way that we can get a positive PDF with the well-known exponential tail. The exponent of the algebraic tail is derived to be $-5/2$ in any dimension. Finally we observe that various forcing spectrums do not alter the power law behaviour of the algebraic tail in these dimensions, due to a relation between forcing correlator exponent and the exponent of the two-point density correlation function.
PACS numbers 47.27.Gs and 47.40.Ki

The interest in solving the randomly driven Burgers equation with a large scale driving force is motivated by the hope that it can provide us with the first solvable model of turbulence. Consequently tremendous activity emerged on the non-perturbative understanding of Burgers turbulence [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. As an original attempt, Polyakov offered a field theoretical method to derive the probability distribution or density function (PDF) of velocity difference in the problem of randomly driven Burgers equation in one dimension [1]. The problem of computation of correlation functions in the inertial range is reduced to the solution of a closed partial differential equation [1, 2]. However, Polyakov's approach was based on the conjecture of the existence of the operator product expansion (OPE). This method was then extended to the forced Burgers equation coupled to the continuity equation and some results were derived for the behaviour of the probability density function tails and for the value of intermittency and density-density correlators exponents [3, 12]. On the other hand, some extensive numerical simulations show that the

predictions of these theoretical works coincide with the numerical simulations, within a good approximation [10, 13].

It was known [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13] that the PDF in Burgers turbulence behaves physically different in two different intervals of velocity difference u. When $|u| > U_{rms}$ the PDF behaves as:

$$P(u,r) = rG(\frac{u}{U_{rms}})$$

which depends on the single-point property U_{rms} (root-mean-square of velocity fluctuations) and therefore is not a universal function. While in the interval $u \ll U_{rms}$ and $r \ll L$ (where L is the energy input scale), the PDF can be represented in the universal scaling form:

$$P(u,r) = \frac{1}{r^{\delta}} F(\frac{u}{r^{\delta}})$$

where $\int_{-\infty}^{+\infty} F(x)dx = 1$ and the exponent δ is related to the random force spectrum [13]. Results for one-dimensional case indicates that the function $F(x)$ behaves like $\exp[-c(\delta)x^3]$ when $x \to +\infty$, where the coefficient $c(\delta)$ can be evaluated from the theory [8, 13]. In this region, various algebraic behaviours were predicted for the PDF in the limit $x \to -\infty$ and determination of the asymptotic behaviour of the PDF is at the moment controversial. Several different proposals have been made, each leads to an asymptotic experssion of the forms $\sim |u|^{-\alpha}$ but with a variety of values for the exponent α includes 2 [6, 13], the range 5/2 to 3 [1, 2, 12], 3 [10] and 7/2 [7, 8, 9]. Numerical simulations performed in [13] shows that the power law exponent of the PDF depends on the forcing spectrum. The investigations of forced Burgers turbulence models have also given furthur understanding of intermittency in the velocity structure functions. It is believed that intermittency in these systems is a consequence of the algebraic tail of PDF implying the scaling exponent of velocity structure functions, ξ_n, is saturated to the value $\xi_n = 1$ for $n > n_c$, where the value of critical moment number n_c depends on the forcing function spectrum [13]. This simply means that the moments $S_{n>n_c}$ are dominated by the non-universal region and thus vary with the sigle-point property U_{rms} induced by the large scale random force. At the same time, the moments with $n < n_c$ are universal [1, 13].

The first attempt on d-dimensional Navier-Stokes turbulence was made in [15] as a renormalized perturbation expansion with $1/d$ as a small parameter. This attempt failed because $1/d$ factors appearing in the expansion are cancelled by the $O(d)$ multipliers resulting from the summation over the d components of velocity field. The problem of d-dimensional turbulence was recently revisited [16] using Polyakov's non-perturbative approach. It was shown that in the limit $d \to \infty$ Kolmogorov scaling is an exact solution of the incompressible Navier-Stokes equations and the pressure contributions to the equations are responsible for a clear distinction between the Navier-Stokes and Burgers dynamics. In an other work [3], the asymptotics of the PDF of velocity gradient were found in d-dimensional randomly driven Burgers equation in a compressible fluid.

In the present paper we investigate the isotropic forced Burgers equation coupled to the continuity equation in arbitrary dimensions. In the inviscid limit, we find a complete closed equation for the generating function of velocity difference. We solve this equation for the longitudinal component of velocity difference in the universal region and show that the main requirments on the PDF fix the exponent of two-point density correlation function. This results in a dimension dependent exponent for density fluctuations. This then means that the equation governing the PDF is independent of dimensions. In this region, our results predict the exponential decaying of the right tail of PDF of velocity difference which is also obtained in several other works [1, 2, 3, 4, 6, 8, 9, 10, 12]. Thus the power law decay of the left tail is independent of the dimensionality and the forcing spectrum with the exponent $-5/2$ in any dimension. While the previous results for one-dimensional forced Burgers turbulence exhibit a force spectrum dependent tail [1, 2, 13] or a different power law decaying [6, 7, 8, 9, 10]. Finally, we discuss Polyakov's approach [1] for dealing with the problem in the limit of infinitesimal viscosity. For a special forcing spectrum this method would not give anything more than the inviscid calculations. Our work extends a previous result [12] on two and three-dimensional Burgers turbulence to higher dimensions.

Let us start with the Burgers and continuity equations:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{x},t) \tag{1}$$

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{2}$$

for the irrotational velocity field $\mathbf{u}(\mathbf{x},t)$, viscosity ν and density ρ in d dimensions. The force $\mathbf{f}(\mathbf{x},t)$ is the external stirring force, which injects energy into the system on a length scale L. More specifically, one can take, for instance a Gaussian distributed random force, which is identified by its two moments:

$$\begin{aligned}\langle f_\mu(\mathbf{x},t)\rangle &= 0\\ \langle f_\mu(\mathbf{x},t) f_\nu(\mathbf{x}',t')\rangle &= \delta(t-t')k_{\mu\nu}(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{3}$$

where $\mu,\nu = 1,2,\ldots,d$ and the correlation function $k_{\mu\nu}(r)$ is concentrated at some large scale L. The problem is to understand the statistical properties of the velocity and density fields which are the solutions of equations (1) and (2). Following Polyakov, we consider the following two-point generating functional:

$$F_2(\lambda_1,\lambda_2,\mathbf{x}_1,\mathbf{x}_2,t) = \langle \rho(\mathbf{x}_1,t)\rho(\mathbf{x}_2,t)\exp(\lambda_1\cdot\mathbf{u}(\mathbf{x}_1,t)+\lambda_2\cdot\mathbf{u}(\mathbf{x}_2,t))\rangle \tag{4}$$

where the symbol $\langle\ldots\rangle$ means an average over various realization of the random force. Now one can show that F_2 satisfies the following equation:

$$\partial_t F_2 + \sum_{i=1,2;\mu=1,\ldots,d} \frac{\partial}{\partial \lambda_{\mu,i}} \partial_{\mu,i} F_2 - \sum_{i,j=1,2;\mu,\nu=1,\ldots,d} \lambda_{\mu,i}\lambda_{\nu,j} k_{\mu\nu}(\mathbf{x}_i - \mathbf{x}_j) F_2 = D_2 \tag{5}$$

where the first two terms on the left-hand side of equation (5) come from the terms on the left-hand side of equations (1) and (2), and the third is the contribution of forcing term, in which we have used Furutsu-Novikov-Donsker formula [17]. Also, D_2-term is the contribution of dissipation. D_2 is the anomaly term and has the following form:

$$D_2 = \langle \nu \rho(\mathbf{x}_1)\rho(\mathbf{x}_2)[\lambda_1 \cdot \nabla^2 \mathbf{u}(\mathbf{x}_1) + \lambda_2 \cdot \nabla^2 \mathbf{u}(\mathbf{x}_2)] \exp(\lambda_1 \cdot \mathbf{u}(\mathbf{x}_1) + \lambda_2 \cdot \mathbf{u}(\mathbf{x}_2)) \rangle \tag{6}$$

It is noted that although the advection contribution are accurately accounted for in this equation, it is not closed due to the dissipation term. In what follows, we consider the inviscid ($\nu = 0$) Burgers equation to avoid the anomaly problem and leave Polyakov's approach for dealing with the problem in the limit $\nu \to 0$ to the last part of the present paper. Now we change the variables as $\mathbf{x}_\pm = \mathbf{x}_1 \pm \mathbf{x}_2$, $\lambda_+ = \lambda_1 + \lambda_2$ and $\lambda_- = \frac{\lambda_1 - \lambda_2}{2}$. By the Galilean invariance and the spatial homogeneity assumptions the variables $\mathbf{x}_+$ and λ_+ can be set to zero. It is found from the equation (5) that in the *stationary state* we have:

$$\sum_{\mu=1,\ldots,d} \frac{\partial}{\partial \lambda_\mu} \frac{\partial}{\partial x_\mu} F_2 + 2 \sum_{\mu,\nu=1,\ldots,d} \lambda_\mu \lambda_\nu [k_{\mu\nu}(\mathbf{x}) - k_{\mu\nu}(0)] F_2 = 0 \tag{7}$$

where we have used $\mathbf{x}$ and λ instead of $\mathbf{x}_-$ and λ_- for simplification. Also, because of isotropy F_2 can depend only on the absolute values of vectors $\mathbf{x}$ and λ and the angle θ between them as $F_2 = F_2(r, \lambda, s)$ where $r = |\mathbf{x}|$, $\lambda = |\lambda|$ and $s = \cos\theta = (\sum x_\mu \lambda_\mu)/r\lambda$. We suppose the stirring correlation function has the following form:

$$k_{\mu\nu}(\mathbf{x}) = k_0 \delta_{\mu\nu} - \frac{k_1}{2L^2} r^2 (\delta_{\mu\nu} + 2\frac{x_\mu x_\nu}{r^2}) \tag{8}$$

with k_1 and $L = 1$. Now, using spherical coordinates (r, λ, s) it can be shown from equations (7) and (8) that F_2 satisfies the following closed equation for homogeneous and isotropic turbulence:

$$[s\partial_r\partial_\lambda - \frac{s(1-s^2)}{r\lambda}\partial_s^2 + \frac{(d-2+s^2)}{r\lambda}\partial_s + \frac{(1-s^2)}{\lambda}\partial_r\partial_s + \frac{(1-s^2)}{r}\partial_\lambda\partial_s - r^2\lambda^2(1+2s^2)]F_2 = 0 \tag{9}$$

The one-dimensional case of equation (7) is easily recovered by setting $s = 1$. We wish to consider the longitudinal velocity component statistics which results from equation (9) by taking the limit $s \to 1$. Assuming F_2 be a regular function near $s = 1$ we can

safely drop the terms multiply by $(1-s^2)$ in equation (9). We propose the universal scale invariant solution of equation (9) in the following form:

$$F_2(r,\lambda,s) = g(r)F(\lambda r^{\delta},s)$$

$$g(r) = r^{-\alpha_d} \tag{10}$$

In equation (10), $g(r)$ is the conditional two-point correlation function of density field conditioned on a fixed value of velocity differnce. We assume that $g(r)$ depends only on r and its dependence on the velocity interval appears in the exponent α_d, we shall discuss this further in the next section. For a general stirring correlation function $k_{\mu\nu}{\sim}(1-r^{\zeta})$, the exponent δ is found by substituting the proposed form of generating function as $\delta = \frac{\zeta+1}{3}$ which in our case ($\zeta = 2$) is $\delta = 1$. Indeed, we assume that the two-point density correlation function exists, and therefore it is necessary to find $F(\lambda r^{\delta}, s)$ such that it tends to a constant in the limit of $\lambda \to 0$. Also, It is straight forward to show that the mean value of velocity difference is zero as expected from the homogeneity and isotropy constraints. Now let us consider the longitudinal velocity component statistics suggested by equation (9) in the limit $s \to 1$. In this limit, we assume that the generating function of velocity difference $F(\lambda r, s)$ has the following form:

$$F(\lambda r, s) = F(\lambda r s) \tag{11}$$

this form ensures the factorizing property of the angular part of structure functions as $S_n(r,s) \sim s^n S_n(r)$ when $n < 1$. The factorization of the angular part of velocity structure functions in the limit $s \to 1$ has also been known for the Navier-Stokes turbulence [14]. Plugging the ansatz (10) and (11) into equation (9) and rewriting it for the variable $z = \lambda r s$ gives the following equation for the generating function of longitudinal velocity difference:

$$zF''(z) + (d-\alpha_d)F'(z) - 3z^2F(z) = 0 \tag{12}$$

It is interesting that equation (12) is similar to the equation first derived by Polyakov [1] for the problem of one dimensional Burgers. In that work the effect of the viscous term is found by taking the limit of infinitesimal viscosity applying the self-consistent conjectures of operator product expansion. The anomaly terms which arise from the viscous term modify the master equation of the generating function in such a way that a positive, finite and normalizable PDF can be found. Comparing equation (12) with Polyakov's result shows that the anomaly term in Polyakov's approach is replaced with the exponent of two-point density correlation function such that a simple change of the parameters maps equation (12) to one derived by Polyakov [1]. Therefore, as mentioned in [12], the presented approach for inviscid forced Burgers turbulence shows that considering the density fluctuations coupled to the velocity field, alters the governing equations in such a way that we can obtain positive PDF

even in the inviscid limit and with no need for the viscosity anomaly. It is easy to show using equation (12), as discussed in [1], that the main requirments on the PDF forces us to take the two point density correlator exponents in arbitrary dimensions as:

$$\alpha_d = d + \frac{1}{2} \tag{13}$$

which yields $F(z) = \exp[\frac{2}{\sqrt{3}}(z^{3/2})]$. This result consequently gives the right tail of the PDF of longitudinal velocity differrence as $\frac{1}{r}\exp[-c(\frac{u}{r})^3]$ (for $\frac{u}{r} \to +\infty$) where $c = \frac{1}{9}$. This tail coincides with the result of several other approaches [1, 2, 3, 4, 6, 8, 9, 10]. The left tail of the PDF is obtained in the limit of $\frac{u}{r} \to -\infty$ as $|u|^{-(\alpha_d - d + 2)}$ or, by equation (13), $|u|^{-5/2}$. As a result, density fluctuations which couple to the velocity field appear themselves in the exponent of left tail of PDF and lead to a dimensionality independent decay law.

Now we wish to focus our attention on the dependnce of these results on the stirring force spectrum. We consider a general large-scale stirring force correlator as follows:

$$k_{\mu\nu}(\mathbf{x}) = k_0\delta_{\mu\nu} - \frac{1}{2}r^{\zeta}(\delta_{\mu\nu} + 2\frac{x_\mu x_\nu}{r^2}) \tag{14}$$

With this choice the velocity field remains irrotational also the coefficients in expression (14) are taken such it simply reduces to equation (8) when $\zeta = 2$. The last term in the equation (5) can be written in the form $-ar^{\zeta}\lambda^2 F_2$ for a general form of the stirring force. The coefficient a depends on the parameters in the expression of the stirring correlator. For the choice as in equation (14) a is $(1+2s^2)$. However the exact form of a does not affect the main features of the problem such as algebraic tail of the PDF. Taking the universal solution (10) for the new master equation gives the following eqation for the generating function $F(\lambda r^{\delta} s) = F(z)$ in the limit $s \to 1$:

$$\delta z F''(z) + (d - \alpha_d + \delta - 1)F'(z) - 3z^2 F(z) = 0 \tag{15}$$

where $\delta = \frac{1+\zeta}{3}$. The PDF of longitudinal velocity difference is the inverse Laplace transform of $F(z)$. The positivity, finitness and normalizability requirements on the PDF will fix the exponent of two-point density correlator as:

$$\alpha_d = d + \frac{1}{2}(\zeta - 1) \tag{16}$$

also we obtain $F(z) = \exp[\frac{2}{\sqrt{3\delta}}(z^{3/2})]$. The right tail of the PDF can be readily deduced as $\exp[-cx^3]$ when $x = \frac{u}{r^{\delta}} \to +\infty$ and $c = -\frac{\delta}{9}$. On the other hand, the limit $x \to -\infty$ gives the left tail in the following form:

$$P(u,r) \sim |u|^{-(\alpha_d - d + \delta + 1)/\delta} \tag{17}$$

Substituting the value of α_d from equation (16) and $\delta = \frac{\zeta+1}{3}$, we obtain the algebraic tail as:

$$P(u,r) \sim r^{3\delta/2}|u|^{-5/2} \tag{18}$$

We observe that the density fluctuations affect the algebraic tail of the PDF and modify its decaying exponent such that it becomes independent of the dimension and the forcing spectrum.

As mentioned previously, in the limit $\nu \to 0$, Polyakov formulated a method for analyzing the inertial range statistics based on the conjecture of the existence of OPE. Extending the assumptions of OPE to take into account the anomaly term in the case of arbitrary dimensions for Burgers equation coupled to the continuity equation, it was shown that D_2-term has the following structure [3]:

$$D_2 = aF_2 \tag{19}$$

where a is generally a function of λ_1 and λ_2. Therefore keeping the viscosity infinitesimal but nonzero produces a finite effect and a new term on the right hand side of equation (9) as:

$$\begin{aligned}[s\partial_r\partial_\lambda - \frac{s(1-s^2)}{r\lambda}\partial_s^2 + \frac{(d-2+s^2)}{r\lambda}\partial_s + \frac{(1-s^2)}{\lambda}\partial_r\partial_s \\ + \frac{(1-s^2)}{r}\partial_\lambda\partial_s - r^2\lambda^2(1+2s^2)]F_2 = a(\lambda)F_2\end{aligned} \tag{20}$$

The λ dependence of $a(\lambda)$ anomaly must be chosen to conform the scaling and can be changed depending on the properties of the force correlation function. For a general correlation function $k_{\mu\nu}{\sim}(1-r^\zeta)$ the λ dependence of a is fixed as $a_0\lambda^\sigma$ where $\sigma = \frac{2-\zeta}{1+\zeta}$ and a_0 is a constant. It is evident that in the case of $\zeta = 2$, $a(\lambda)$ is independent of λ. In this case, one can easily show from the master equation that the parameter a_0 depends linearly on the mean value of velocity difference field of flow and therefore vanishes in the homogeneous isotropic turbulence. However for different types of correlations of the stirring force, e.g.$k_{\mu\nu}{\sim}(1-r^\zeta)$ with $\zeta \neq 2$, we have to assume non-zero a_0 [2]. Thus, for the stirring correlation as the type $1-r^2$, the results of Polyakov's formalism (in the limit of infinitesimal viscosity) will not be different from the inviscid results of homogeneous isotropic turbulence.

To our knowledge, there is a little informations about the statistics of the density field. There exists one simulation [18] to find the PDF tails of density field for one dimensional decaying Burgers turbulence in the zero viscosity limit, which exhibits a power law tail for the density PDF in the high density regime. In one dimension,

Boldyrev [3] has reported a simulation in which the exponent of two-point density correlation function is predicted to be ~ 2.

We would like to thank J. Davoudi, S. Moghimi-Araghi and M.R. Rahimi Tabar for useful discussions.

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